Report

Hagen-Poiseuille flow linear instability

Sergey G. Chefranov[1*], Alexander G. Chefranov[2]

[1] Obukhov Institute of Atmospheric Physics of RAS, 119017, Russia, Moscow, Pijevskii str.3

[2] Eastern Mediterranean University, Famagusta T.R. North Cyprus via Mersin 10, Turkey

[*] Corresponding author phone: +7 (916) 947-63-58, e-mail: schefranov@mail.ru

Submitted: 21 July 2014
Accepted: 15 August 2014
Published online: 14 November 2014

Abstract

In the suggested here linear theory of hydrodynamic instability of the Hagen - Poiseuille flow it is counted the possibility of quasi periodic longitudinal variations, when there is no separation of the longitudinal and radial variables in the description of the disturbances field. It is proposed to use the energetic method and the Galerkin approximation method that takes into account existence of different values of longitudinal variability periods for different radial modes corresponding to the equation of evolution of extremely small axially symmetric velocity field tangential component disturbances and boundary condition on the tube surface and axis. We found that even for two linearly interacting radial modes the HP flow may have linear instability, when $\mathrm{Re} > \mathrm{Re}_{th}(p)$ and the value $\mathrm{Re}_{th}(p)$ very sensitively depends on the ratio *p* of two longitudinal periods each of which describes longitudinal variability for its own radial mode only. Obtained from energetic method for the HP flow linear instability realization minimal value $\mathrm{Re}_{th\min} \approx 704$ (for N=600 radial modes) and from Galerkin approximation $\mathrm{Re}_{th\min} \approx 448$ (for N=2 modes with p=1.516) which quantitatively agrees with the Tolmin-Shlihting waves in the boundary layer arising, where also the threshold value $\mathrm{Re}_{th} = 420$ is obtained. We state also the agreement of the phase velocity values of the considered in our theory vortex disturbances with the experimental data on the fore and rear fronts of the turbulent "puffs" spreading along the pipe axis.

PACS: 47.20.Ft, 47.27.Cn, 47.27.nf

Keywords: Linear hydrodynamic instability • Spiral vortex flow • Pipe flow

Imprint: Sergey G. Chefranov, Alexander G. Chefranov. Hagen-Poiseuille flow linear instability; Cardiometry; No.5; November 2014; p.17-34; doi:10.12710/cardiometry.2014.5.1734
Available from: www.cardiometry.net/no5-november-2014/flow-linear-instability

Introduction

Fundamental and applied problem of defining of the turbulence arising mechanism for the Hagen-Poiseuille (HP)[1] flow more than century is left mysterious because of the linear stability paradox of the flow with respect to extremely small by amplitude disturbances for any Reynolds number value $Re = \frac{V_{max} R}{\nu}$ (where V_{max}, ν, R are the maximal HP flow near axis velocity, kinematic viscosity coefficient, and pipe radius respectively) [1-4]. Obvious contradiction with experiments corresponding to the paradox now is used to be coped with based on an assumption of permissibility of the HP flow instability with respect to disturbances having sufficiently large finite amplitude strict non-linear mechanism only [5-10]. The basis for such the assumption (see [3, 4]) gives one side interpretation of experiments [11] in which many-fold increase of the threshold Reynolds number value Re_{th} up to 100000 is achieved due to the increase of the level of smoothness of the streamlined pipe surface. In this interpretation, only correlation between the surface smoothness increase and resultant decrease of the average amplitude of the original disturbances is taken into account. At the same time, noted even by O. Reynolds [1] extremely high sensitivity of the value of Re_{th} to the initial disturbance does not exclude possibility of impact on Re_{th} of not only amplitude but also space-time characteristics of the disturbances also caused by non-ideal smoothness of the streamlined surface. Actually, for example in the experiment [12], it is found that under the fixed amplitude of artificially excited disturbances, instability of the HP flow emerges only in some definite narrow range of the disturbances' frequencies.

In the present work, we show that possibility of linear absolute (i.e., non-convective [4]) instability of the HP flow is defined by the value of complementary to the Reynolds number *Re* control parameter p, which characterizes frequency-wave features of the disturbances and determines the value of the threshold Reynolds number $Re_{th}(p)$ independently from the amplitude of the initial disturbances. Such complementary parameters are easily introduced in all known HP flow modifications – in the cases of the flow in the pipe with the existence of near-axis cylinder [13], in the pipe with elliptic cross section [14], in the rotating pipe [15, 16], and even for a flow transferring particles of finite size in a pipe [17]. In all these examples, there already exists complementary to the Reynolds number control parameter p and the linear stability theory paradox is absent. This examples show that "circumvention" (see [5]) of the HP flow linear stability paradox due to the consideration of strict finite amplitude only mechanism of instability of the flow "hardly can satisfy anybody" [18].

Introduction of such a complementary parameter p for the HP flow is already not as obvious as for the HP flow modifications in [13–17]. It however is performed below on the base of pointed by O. Reynolds [1] (and then by W. Heisenberg also for the flat Poiseuille flow, see in [4, 6]) concept of dissipative instability mechanism[2] of the HP flow related with the action of molecular viscosity ν near the very solid boundary. According to [1], the mechanism manifests itself in the form of

[1] HP flow is by definition a laminar stationary flow of the uniform viscous fluid along the static straight linear and unbounded in length pipe with round same along the whole pipe axis cross section

[2] Such a mechanism is naturally realized in the systems having disturbances with negative energy [19-22], for example, for threshold emergence of vortexes (rotons) in the flow of super-fluid helium in a capillary [19].

spontaneous one-step emergence for $\mathrm{Re} > \mathrm{Re}_{th}$ of vortexes having character size L_ν, «..that is already not growing as it was expected with the growth of the velocity amplitude [1]». That is why, the value L_ν must significantly differ also from the length scale $l_\nu = \nu / V_{\max}$ that leads to the Reynolds number defining as $\mathrm{Re} = R / l_\nu$ and explicitly depending on the stream velocity maximal amplitude value. Such scale L_ν seemingly related also with the level of the streamlined pipe surface smoothness, may together with the of radius *R* define not amplitude only but also frequency-wave initial disturbance parameters, for example, their longitudinal along the pipe axis (axis *z*) spatial periods. The ratio of the periods $p = \frac{L_\nu}{R}$ as it is shown below is a new complementary parameter defining the HP flow linear instability threshold with respect to extremely small by amplitude vortex disturbances. Note that for any Reynolds number value, *p* can vary in vast range from *p<<1* to *p>>1*.

It is suggested here to use disturbance structure representation in the form of two radial modes each of which having its own period of longitudinal variability differing from that of the other mode. Such representation corresponds to the observed conditionally periodic Tolmin - Shlihting (TS) waves emergence of which (caused also by near-boundary action of the molecular viscosity) precedes blow-like emergence of the turbulence in the near-boundary layer [23-25]. Besides that, even in [2, 26], it is noted that usually considered in the linear stability theory "normal" periodic by z disturbances fields obviously don't correspond to the structures observed in the experiments, for which different longitudinal periods for different radial modes are characteristic.

In the present work, it is shown that leaving off the assumption of separation of the longitudinal and radial variables defining spatial disturbance variability leads now to the finite value of the minimal threshold Reynolds number $\mathrm{Re}_{th} \approx 448$ ($p \approx 1.53..$). Close to it threshold Reynolds number value is characteristic also for the observed threshold for the transition from the laminar resistance law to another one [2, 27] and for the conditions of excitation of TS waves in a boundary layer [25]. We have conducted comparison of the considered theory with the experimental data for the flow in the pipe [28-30] and also with the conclusions of the stability theory (Tolmin-Shlihting and Lin) and experimental data on the stability of laminar near-boundary layer [31]. We obtained correspondence not only of the quantitative values of the critical Reynolds number for linear exponential instability for the HP flow and for TS waves excitation (where also $\mathrm{Re}_{th} = 420$), but also similar shapes of instability regions (bounded by the curves of neutral stability). This also confirms expected above similarity of their viscous dissipative realization mechanisms.

Materials and methods

1. The statement of the problem

Let us consider known (see [4]) representation of the HP flow in the cylindrical reference frame (z,r,φ): $V_{0r}=V_{0\varphi}=0, V_{0z}=V_{\max}(1-\frac{r^2}{R^2})$, where $V_{\max}=\frac{R^2}{4\rho\nu}\frac{\partial p_0}{\partial z}$, the fluid density $\rho = const$, $\frac{\partial p_0}{\partial z}$ is the constant value of the pressure gradient p_0 along the axis of the pipe of radius R, and ν is the coefficient of kinematic fluid viscosity.

The linear stability of this flow is considered for the more simple case, when there exist only extremely small disturbances of tangential component of velocity and the stability to the "normal" pure periodic longitudinal disturbances is easy to determine at all Reynolds numbers. We demonstrate here that, when instead of this "normal" form, it is taken into consideration quasi periodic longitudinal variability of disturbances, it is possible for HP flow to be linear unstable for the finite Reynolds numbers larger than some threshold value.

In the axially symmetric case (i.e. for extremely small disturbances not depending on the angular coordinate φ) linear instability of the HP flow can be defined by the tangential velocity component V_φ only, which meets the following equation in dimensionless form (when *y=r/R, x=z/R,* $\tau = t\nu/R^2; \mathrm{Re} = V_{\max}R/\nu$) and corresponding boundary condition on the rigid surface of the tube at *y=1* and at the axis of the tube at *y=0*:

$$\frac{\partial V_\varphi}{\partial \tau} + \mathrm{Re}(1-y^2)\frac{\partial V_\varphi}{\partial x} = \Delta V_\varphi - \frac{V_\varphi}{y^2}; \Delta = \frac{1}{y}\frac{\partial}{\partial y}y\frac{\partial}{\partial y} + \frac{\partial^2}{\partial x^2}, \quad V_\varphi(y=1)=0; V_\varphi(y=0)=0 \tag{1}$$

where the value V_φ in (1) is also dimensionless (normalized on $V_{\max}$).

When other components of velocity disturbances are absent in this axially symmetric case the equation and boundary conditions (1) are also useful to describe evolution of finite amplitude disturbances, not only extremely small ones. The solution of equation (1) must be found with boundary condition $V_\varphi = 0$ at *y=0* because the angular velocity of rotation on the axis of the tube must be finite. It is useful to represent the solution of equation (1) satisfying boundary condition of (1) as follows:

$$V_\varphi = e^{\lambda\tau}V; V = \sum_{n=1}^{N} A_n(x) J_1(j_{1,n}y); J_1(j_{1,n}) = 0; \lambda = \lambda_1 + i\lambda_2; A_n = A_{1n} + iA_{2n}, \quad A_n(x) = A_n(x+T_n); T_n = 1/j_{1,n}; \max T_n = 1/j_{1,1} \tag{2}$$

In (2), J_1 - is the Bessel function of the first order and the value N must be considered as infinite for obtaining the exact solution of (1) with mentioned in (1) and (2) boundary conditions on y and x.

Thus, instead of the traditional "normal" periodic form (which coincides with (2), when in (2) $T_n = T = const$ for all n) we introduce in (2) the new definition of individual periodic boundary condition along the infinite tube for each radial mode with number *n=1,2,..,N*.

The statement of the problem (1), (2) for the longitudinal quasi periodic disturbances *n=1, 2,. ..,N,* is different from previously used in linear theory of hydrodynamic stability, where the pure periodic conditions along the tube are considered. In the experimental data [2, 26], from the other side, only quasi periodic variations take place and (2) is better complying to them than traditional "normal" periodic form of disturbances.

2. Energy consideration

Let us consider, on the base of (1), (2), evolution of the average energy (on the unit of mass):

$$E = \langle V_\varphi V_\varphi^* \rangle / 2 = e^{2\lambda_1 \tau} \langle VV^* \rangle / 2;$$
$$I_0 = \langle VV^* \rangle = 2\int_0^1 dyy \frac{1}{T_{\max}} \int_0^{T_{\max}} dx VV^*. \quad (3)$$

For example, in (3), we may take $T_{\max} = 1/ j_{1,1}$.

From (1), it is possible to obtain the following equation for the exponential index λ_1 of energy growth (for $\lambda_1 > 0$) or fall (if $\lambda_1 < 0$) in time:

$$2\lambda_1 I_0 = \operatorname{Re} I_1 - I_2; I_2 = -\left\langle V^*(\Delta V - \frac{V}{y^2}) + V(\Delta V^* - \frac{V^*}{y^2}) \right\rangle$$
$$I_1 = -\left\langle (1-y^2)\frac{\partial (VV^*)}{\partial x} \right\rangle = \frac{1}{T_{\max}} \sum_{n=1}^{N}\sum_{m=1}^{N} q_{nm}(A_n(T_{\max})A_m^*(T_{\max}) - A_n(0)A_m^*(0)) - I_{11},$$
$$I_{11} = \frac{1}{T_{\max}} \sum_{n=1}^{N} J_2^2(j_{1,n})(A_n(T_{\max})A_n^*(T_{\max}) - A_n(0)A_n^*(0)), \quad (4)$$
$$q_{nm} = q_{mn} = 2\int_0^1 dyy^3 J_1(j_{1,n}y)J_1(j_{1,m}y);$$
$$I_2 = \sum_{n=1}^{N} J_2^2(j_{1,n}) \frac{1}{T_{\max}} \int_0^{T_{\max}} dx \left[2j_{1,n}^2 A_n A_n^* - A_n \frac{d^2 A^*}{dx^2} - A_n^* \frac{d^2 A}{dx^2} \right] > 0,$$

where J_2 is the Bessel function of the second order.

Let's consider a special case, when in (2)–(4), $A_n(x) = A_{0n} \exp(i2\pi\alpha_n x + i2\pi\beta_n)$, and, from (4), we have:

$$I_2 = 2\sum_{n=1}^{N} (j_{1,n}^2 + 4\pi^2\alpha_n^2) A_{0n}^2 J_2^2(j_{1,n}); I_0 = \sum_{n=1}^{N} A_{0n}^2 J_2^2(j_{1,n}); \quad (5)$$

$$I_1 = -2\alpha_1 \sum_{n=1}^{N} \sum_{\substack{m=1 \\ m\neq n}}^{N} q_{nm} A_{0n} A_{0m} \sin(\pi|p_n - p_m|) \sin(\pi|p_n - p_m| + 2\pi|\beta_n - \beta_m|) \ , \quad (6)$$

where $p_n = \alpha_n / \alpha_1$ ($p_1 = 1; p_2 = \alpha_2 / \alpha_1 \equiv p; p_3 = \alpha_3 / \alpha_1$ and so on). For the convergence of sums in (5), (6) in the limit of $N \to \infty$, the following inequality for amplitudes shall hold: $A_{0n} < 1 / j_{1,n}^{1+k}, k > 0$.

For $A_{0n} = A_0 / j_{1,n}^{1+k}$ when k>0 and $\beta_n = \beta = const$ for all n, from (5), (6), we can obtain the following criterion of the HP flow linear instability (only for the cases with c>0 in (7)):

$$\mathrm{Re} > \mathrm{Re}_{Eth} = I_2 / I_1 = \frac{a^2 + \alpha_1^2 b^2}{\alpha_1 c}; I_2 / 2A_0^2 = a^2 + \alpha_1^2 b^2; I_1 / 2A_0^2 = \alpha_1 c;$$
$$c = -\sum_{n=1}^{N} \left[\sum_{\substack{m=1 \\ m \neq n}}^{N} \frac{q_{nm} \sin^2(\pi(p_n - p_m))}{j_{1,n}^{1+k} j_{1,m}^{1+k}} \right]; \tag{7}$$

$$a = \left[\sum_{n=1}^{N} \frac{J_2^2(j_{1,n})}{j_{1,n}^{2k}} \right]^{1/2}; b = 2\pi \left[\frac{J_2^2(j_{1,1})}{j_{1,1}^{2+2k}} + p^2 \frac{J_2^2(j_{1,2})}{j_{1,2}^{2+2k}} + \sum_{n=3}^{N} \frac{p_n^2 J_2^2(j_{1,n})}{j_{1,n}^{2+2k}} \right]^{1/2}. \tag{8}$$

Minimization of $\mathrm{Re}_{Eth}(\alpha_1; p; p_3; \ldots; p_N)$ in (7), (8) on α_1 gives the following criterion of the HP flow linear instability (when $\alpha_1 = \alpha_{1\min} = a / b$):

$$\mathrm{Re} > \mathrm{Re}_{\substack{Eth\min \\ \alpha_1 = \alpha_{1\min}}} = \frac{2ab}{c}. \tag{9}$$

In (9), for the limit of $N \to \infty$, if $p_n = \alpha_n / \alpha_1 = j_{1,n} / j_{1,1}$, *n=3,4,...,* there is only one free continuous parameter *p* on which $\mathrm{Re}_{\substack{Eth\min \\ \alpha_1 = \alpha_{1\min}}}(p)$ can be minimized. It allows determining of the threshold Reynolds number absolute minimum $\mathrm{Re}_{thabs\min} = 704$ when *p=0.481* and *k=0.7* (minimization was made over *p* and *k*). Dependence of the right hand side of (9) on *p* is presented in Fig.1d for *N=600*, *k*=0.7.

Thus, only when $\nu \neq 0$ in (1), it is possible to expect realization of the HP flow viscous dissipative instability for some above threshold Reynolds numbers, $Re > \mathrm{Re}_{th}$. Actually, when $\nu = 0$, the right-hand side of (1) also turns to zero (if considering (1) in its original dimensional form). In that case, only convective disturbance transfer without change of its form and amplitude in time takes place. From (4), it follows that instability of the HP flow obviously is not realizable also in the case of the pure periodic variability of V_φ along the pipe (see also (24.7) in [2]) when $I_1 = 0$ and $\lambda_1 < 0$ in (4). If to refuse from that, so called "normal", periodic form for the disturbance field in (1), as we do in (2), it is possible to obtain another result from (4) allowing to realize the conditions, when in (6)-(8), the value of I_1 is positive, $I_1 > 0$, and, under condition (9), exponential growth of the disturbance energy with $\lambda_1 > 0$ takes place.

Let's also note that due to the consideration in (1) of only velocity field tangential component disturbances, conservation of the mass stream through the cross section of the pipe for the superposition of the main flow and disturbance field is provided identically. Natural emergence of such disturbances in an axially symmetrical stream really may be hindered although it can't be fully excluded due to the possibility of presence of corresponding randomly-non-uniform smoothness of the streamlined pipe surface. In the laboratory simulation of the HP flow such disturbances can be easily artificially created (see [12]). Let's note also that combination of the main stream and considered disturbance field has non zero value of integral helicity.

For the disturbance representation in the form (2), on the boundary at *y=1*, only vortex field radial component $\omega_y = -\frac{\partial V_\varphi}{\partial x}$ shall turn to zero. The value of the longitudinal vortex field component $\omega_x = \frac{1}{y}\frac{\partial (yV_\varphi)}{\partial y}$ on the boundary at *y=1* has already non zero value that corresponds to the character of forming of the vortex disturbances due to interaction of the stream with the solid pipe wall caused by viscosity forces.

In the considered energy theory, we have used for amplitudes A_n, characterizing different radial modes, only restrictions related with the necessity of convergence of the sum I_2 in (5).

3. The Galerkin–Kantorovich and Bubnov-Galerkin methods

On the base of using the Galerkin–Kantorovich method, for the coefficients A_n, characterizing amplitudes of the linearly interacting disturbance field radial modes, from (1), (2), we get the following system of equations in dimensionless form:

$$\frac{\partial A_m}{\partial \tau} + j_{1,m}^2 A_m - \frac{\partial^2 A_m}{\partial x^2} + \mathrm{Re}\sum_{n=1}^{N} P_{nm}\frac{\partial A_n}{\partial x} = 0, \quad (10)$$

where *m=1, 2, 3, ...* . In (10), constant coefficients P_{nm} have the form

$$P_{nm} = \frac{2}{J_2^2(j_{1,m})} Q_{nm}; Q_{nm} = \int_0^1 dy\, y(1-y^2) J_1(j_{1,n}y) J_1(j_{1,m}y) , \quad (11)$$

where J_2 are the Bessel functions of the second order and the linear with respect to y term under the integral sign yields in P_{nm} the contribution in the form of unity matrix $\delta_{nm} = \begin{cases} 1, n=m \\ 0, n \neq m \end{cases}$. For $N=1$ in (10), the last term can be excluded by Galileo transformation and hence for $N=1$, there is no opportunity of the global absolute instability of the HP flow. In that relation, we shall consider (10) in the simplest non-trivial case $N=2$, that allows already to resolve the HP flow linear stability paradox and leads to the conclusions quantitatively agreeing with the experimental data [29, 31].

As it was already noted, observed in the experiments field structures do not correspond to strictly periodic along the pipe axis disturbances changes (see above and [2, 26]). More over, in [26], it is noted that different radial modes (defining dependence of the disturbances on the radial coordinate) have corresponding differing each from the other variability periods along the pipe axis. This behavior of the observed disturbances change can be modeled with the help of the use in the representation of the system (10) solution an assumption on the difference of the longitudinal periods along the pipe axis for radial modes with different values of index *m*. Such a requirement corresponds to the introduction for each of these modes of its own, independent from the other modes, periodical boundary condition on *x*. In the result, there emerges necessity in the use of the adequate to the pointed boundary conditions Galerkin's approximation of the system (10) solution. Let in (10), for $N=2$, amplitudes A_1 and A_2 have the form of the running waves with different periods along the pipe axis:

$$A_1 = \sum_{n=1}^{M} A_{n10} e^{\lambda\tau + ix2\pi\alpha n},\ A_2 = \sum_{n=1}^{M} A_{n20} e^{\lambda\tau + ix2\pi\beta n}, \tag{12}$$

where A_{n10} and A_{n20} are the constant values. Meanwhile, complementary to the Reynolds number Re control parameter can be defined as $p=\frac{\alpha}{\beta}$ for any α and β. Using (12), from (10) when *N=2*, we get on the base of using of Bubnov-Galerkin weighed differences method the following system for *2M* unknown constant coefficients appearing in (12) under the symbols of summation:

$$A_{m10}(\lambda + j_{1,1}^2 + 4\pi^2\alpha^2 m^2 + i2\pi\alpha m P_{11}\mathrm{Re}) + \sum_{n=1}^{M} i2\pi\beta n P_{21}\mathrm{Re}\, I_1(n,m) A_{n20} = 0;$$

$$\sum_{n=1}^{M} i2\pi\alpha n P_{12} I_2(n,m)\mathrm{Re}\, A_{n10} + A_{m20}(\lambda + j_{1,2}^2 + 4\pi^2\beta^2 m^2 + i2\pi\beta m P_{22}\mathrm{Re}) = 0; \tag{13}$$

$$I_1 = \alpha\int_0^{1/\alpha} dx \exp(i2\pi x(\beta-\alpha)); I_2 = \beta\int_0^{1/\beta} dx \exp(i2\pi x(\alpha-\beta)); I_1 I_2 = -\frac{p\sin(\pi p)\sin(\pi/p)e^{i\pi(p+1/p)}}{\pi^2(1-p)^2},$$

$$I_1 = \alpha\int_0^{1/\alpha} dx \exp(i2\pi x(\beta-\alpha)) = -\frac{i\alpha}{2\pi(\beta-\alpha)}(\exp(i2\pi(\frac{\beta}{\alpha}-1))-1);$$

$$I_2 = \beta\int_0^{1/\beta} dx \exp(i2\pi x(\alpha-\beta)) = -\frac{i\beta}{2\pi(\alpha-\beta)}(\exp(i2\pi(\frac{\alpha}{\beta}-1))-1);$$

$$I_1 I_2 = -\frac{p\sin(\pi p)\sin(\pi/p)e^{i\pi(p+1/p)}}{\pi^2(1-p)^2},$$

For simplicity, let's consider the case *M=1* in system (12) (i.e., farther, we shall use $A_{110} \equiv A_{10}; A_{120} \equiv A_{20}$).

In the result, the system (13) is transformed into a uniform system with constant coefficients for the unknown values A_{10} and A_{20}. From the condition of solvability of this system for the non zero A_{10} and A_{20}, we define the value of the exponent $\lambda = \lambda_1 + i\lambda_2$ depending on dimensionless parameters Re, p and β (see (A.1) and (A.2) in Appendix).

It is interesting also to consider more general cases with *N>2* and *M>1*. Here we note only permissibility of the very fact of existence of linear exponential (not algebraic, with the power law of growth with time) instability of the HP flow stated in the present work. Let's note that even if in the system (10), to turn the kinematic viscosity coefficient to zero, then this does not exclude as for (1) possibility of the HP flow linear instability realization. It is related with the fact that already the very inference of (10) from (1) and (2) for *N>1* is based on the finiteness of the kinematic viscosity coefficient in (1).

The condition of existence of linear exponential instability has the form (A.3). For $\mathrm{Re} >> 1$, (A.3) may be reduced to (A.4). Meanwhile, the value β, defined in (A.5), minimizes the expression for Re_{th} in (A.4) and defines the following condition of the HP flow linear instability condition (giving the estimate from below of the exact value Re_{th}, defined from (A.3)):

$$\mathrm{Re} > \mathrm{Re}_{th} = \frac{\pi^2 (1-p)^2 F^{1/2}}{P_{12} P_{21} p^2 |S|}, \tag{14}$$

where: $S = \sin \pi p \sin \frac{\pi}{p} \sin \pi(\frac{1}{p} + p)$, $B = \frac{S}{|S|}(pP_{11} - P_{22})$,

$$F = (j_{1,2}^2 + j_{1,1}^2)(1+p^2)A^2 + (j_{1,2}^2 - j_{1,1}^2)(1-p^2)B^2 + 2AB(j_{1,2}^2 - p^2 j_{1,1}^2),$$

$$A^2 = B^2 - \frac{4S\, P_{12} P_{21} p^2 ctg\pi(p + \frac{1}{p})}{\pi^2 (1-p)^2}$$ for $P_{11}, P_{22}, P_{12}, P_{21}$ from (11),

because for any p, inequality $A^2 > 0$ holds.

In the condition (14), providing realization of the HP flow linear instability realization Reynolds number threshold value can tend to infinity $\mathrm{Re}_{th} \to \infty$ only for such p, for which the denominator in (14) turns to zero. It takes place for $S = 0$, when the value of the ratio of longitudinal periods is equal to one of the following irrational numbers $p = p_k = k, p = p_{1/k} = \frac{1}{k}$, or equals to one of the irrational numbers defined by the following equality $p = p_{\sqrt{k}} = \frac{k+1 \pm \sqrt{(k+1)^2 - 4}}{2}$ for any integer k ($k = 1,2,..$). For p, related to the intervals of variability p between any two neighboring values $p_k, p_{1/k}, p_{\sqrt{k}}$, the value Re_{th} in (14) is a function of p, having one local minimum on each of the mentioned intervals (see Fig. 1,b). And the value of the absolute minimum $\tilde{\mathrm{Re}}_{th}^{\min} \approx 442$ in (14) is reached for $p \approx 1.53..$, close to the value of the "golden" ratio $p_g = \frac{1+\sqrt{5}}{2} \approx 1.618..$ (i.e., the limit of the infinite sequence of the ratios of two neighboring Fibonacci numbers each of which is equal to the sum of two previous numbers: 1, 2, 3, 5, 8, 13, 21, etc.). For the same p, from the exact condition (A.3), we get close value of the absolute minimum $\tilde{\mathrm{Re}}_{th}^{\min} \approx 448$ (see also Table in the Appendix where conclusions on the base of (A.3) and (14) are compared).

Results and Discussion

The comparison with experimental data and results of TWs (travelling waves) numerical simulation

1. Found value $\widetilde{\mathrm{Re}}_{th}^{\min} \approx 448$ corresponds to the interval of values $\mathrm{Re} \in 300 \div 500$, noted in experimental observation of the threshold transition of the laminar resistance law (for a flow in the pipe) to another already non laminar (but yet not obviously turbulent) resistance mode [2, 27] and for Tolmin-Shlihting (TS) waves in the near wall region of the boundary layer [25]. Observed in [1] and other experiments (see references in [29, 30]) unusual sensitivity of the value Re_{th} to the initial disturbances, actually, corresponds to the obtained in (14) dependency of Re_{th} on p, when, for example, Re_{th} in (14) changes nearly 600 times only when p changes from the value 0,12 to the value 0,11. Neighboring local minima of Re_{th} in (14) also may significantly differ each from the other. For example, for the value $p \approx 2.23$, we have in (14) $\mathrm{Re}_{th} \approx 1982$, and for the value $p \approx 3.86$, already we get $\mathrm{Re}_{th} \approx 84634$. In the scaled form, fragments of the neutral curve, corresponding to the condition (14) (see Fig. 1b), are given on Fig. 1a) in the form of dependency of the value *1/2p* on *Re*. They are plotted on the taken from the paper [31] Fig.12, on which theoretical (of Lin and Shlihting) neutral curves and respective experimental data, related with the determining instability emergence threshold in a boundary layer, are given. Obvious correspondence of the results following from Fig. 1a) allows us making the conclusion also about similarity of the linear dissipative instability mechanisms realized for the HP flow (when meeting the condition (14) or (A.3)), as well as for Tolmin-Shlihting waves excitation in a boundary layer.

2. Conditionally periodic with respect to *x* structure of the initial disturbances field V_{φ} in the representation of the solution (1) in the form (2), (12) agrees with the observed (in [29]) wave velocity field tangential component disturbances changes along the pipe axis. It is especially obvious near very turbulence dying threshold for $\mathrm{Re} \approx 1750$, when on Fig. 5d in [29], it is possible to recognize pairs of characteristic longitudinal variability periods ratios of which are close to the values $p \approx \frac{8}{5}$ $=1.6$ and $p \approx \frac{13}{8} = 1.625$ which are close to the value of the "golden" value of the periods ratio p_g $=1.618...$.

3. On the Fig. 2a) (where Fig. 4 from [30] is used as a basic), dependency on Re of the turbulent spot rear front constant velocity observed for the flow in a pipe is given as well as respective experimental data from [28], scaled by the average flow velocity. There, data are given from [28], corresponding to the turbulent spot rear front velocity description (blue triangles), as well as that of the fore front (light triangles). Also, there, results are given following from the TW in the pipe non-

linear theory [8], and also conclusions of the present work for the phase velocity $V_\beta = -\frac{\lambda_2 V_{max}}{2\pi\beta \mathrm{Re}}$ (scaled by the HP flow average velocity: $V_{av} = \frac{V_{max}}{2}$). The value of the phase velocity is defined in (A.6) from λ_2 in (A.2) for the neutral curve (i.e. under condition $\lambda_1 = 0$). In the present theory, the estimates *V=1.4* and *V=0.8* for the velocities of the fore (leaving the average stream behind) and rear (remaining behind the average stream) fronts of the vortex disturbance in the units of the HP flow average velocity. Experimental data [28, 30] give respective values *V = 1.2* and *V= 0.75*, and numerical calculations on the base of the non-linear theory [8] yield the possibility of the change of *V* from *1.55* to *0.95*. In the result, conclusions of the present linear theory lead to the better agreeing with the experimental data compared with those of the non-linear theory [8], especially in the estimate of the rear front velocity which for the present work is $0.2V_{av}$, for the experiment is $0.25V_{av}$, and for the non-linear theory is $0.05V_{av}$.

Thus, conducted comparison of the suggested theory with the observation data and results of the non-linear theory shows that the data and the conclusions of the present HP flow linear instability theory satisfactory quantitatively and qualitatively agree.

Conclusions

The obtained in the present work new conclusion on possibility of the proof of existence of linear instability for the HP flow is based on the analysis of the system (10) inferred from the initial equation (1) for the evolution of the velocity field tangential component disturbances under condition that the right hand side of (1) is non-zero due to the finiteness of the kinematic viscosity coefficient. Otherwise, when the coefficient is equal to zero, from (1), in principle, it is not possible to get evolution equations for linearly interacting each with the other radial modes and come to the conclusion of the HP flow linear instability. That is why, the mechanism of the found out HP flow linear instability can be called dissipative, and the very instability to consider as the dissipative instability.

Earlier, such hydrodynamic instability dissipative mechanism was considered in the works of L. Prandtl (1921-22) when investigating laminar boundary layer stability, and also of W. Heisenberg (1924) and Lin C. C. (1944-45) when establishing flat Poiseuille flow linear instability (see also [23] and references therein). Qualitative explanation of the physical sense and possibility of appearance of dissipative instability in the problems of hydrodynamic stability are discussed in [18] on the base of elementary accounting of the delay effects on example of an oscillator with the friction linear with respect to velocity. Substantial understanding of the phenomenon of the dissipative instability for HP and other flows near solid boundary surface may also be obtained using a method similar to the one suggested by L.D. Landau in [19] for estimation of the critical velocity of motion of the super-fluid liquid in a capillary. In [19], from the condition of negativity of the energy of an elementary vortex disturbance when for the velocities exceeding that critical value due to the viscous interaction of the stream with the capillary wall, there emerges a vortex disturbance (roton) destroying the laminar super-fluid state of the liquid motion. For the HP flow, for example, also it is interesting to conduct similar to [19 – 22] research aiming defining conditions for realization of the dissipative instability related with the threshold character of the emerging vortex disturbance energy becoming negative valued (in an appropriate inertial reference frame) when exceeding some definite critical Reynolds number.

Let us note, however, that in the present work, to get for the HP flow the conclusion on linear instability, accounting of finiteness of the viscosity is important only on the stage of getting, from the equation (1), the system (10) that defines evolution with time and along the pipe axis, for *N>1,* of the linearly interacting radial disturbance modes. Already in (10) it is possible to consider the limit of infinitely large values of *Re*, corresponding to the ideal liquid with zero viscosity. Meanwhile, it is important only to preserve the suggested above consideration of the linear hydrodynamic stability problem for the very case of the boundary conditions individually defined for each of the both considered (for *N=2*) radial disturbance modes. Only in that case, it is preserved the obtained conclusion about possibility of the HP flow exponential instability but now instead of two control parameters, (*Re, p*), defining the instability region (depending on the wave number β), in the limit of zero viscosity, there will be left only the parameter *p*. In the present work, such a limit of zero viscosity for the system (10) was not considered. Such an investigation on the base of the linear system of interacting radial modes (10) for *N=2* may be interesting in relation with available works on simulation of the processes of instability formation in the flow in a pipe based on the use of the concept of ideal (non-viscous) disturbances describing non-linear pair-wise interacting TWs with small but finite amplitude [32-34]. At the same time, we show in the present work that for the finite value of the kinematic viscosity coefficient, consideration of the limit *Re>>1*, yielding the formula (14) for estimation of the minimal threshold Reynolds number gives not large difference in the value of the estimate (since it was obtained the value $\mathrm{Re}_{th} = 442$) from the exact formula (A.3), where $\mathrm{Re}_{th} = 448$.

Let us note that the considered double vortex-wave structure of the spatial disturbance field variability is in the qualitative agreement with the data of laboratory [35] and numeric [36, 37] experiments in the pipe. This is witnessed also by the conducted in the previous paragraph comparison of the conclusions of the present theory with the experimental data and results of numerical modeling of instability development for the flow in the pipe. And in [36], for example, there were obtained estimates of the turbulent spot phase velocity *V=0.9* and *V=1.1* (in the units of the flow in the pipe average velocity), similar to the presented above.

The radial modes have differing each from the other longitudinal variability periods that corresponds to the use of representation (12) for them. And according to (14), linear exponential instability is found to be possible not only for almost all irrational values of ratios of such longitudinal periods *p*, but also for the rational values of p, not coinciding with $p = p_k$ *=k* or $p_{1/k}$ *=1/k* for integer *k (k=1, 2, 3..)*. An exemption from all possible irrational values *p* constitute only defined from (14) irrational numbers $p = p_{\sqrt{k}} = \frac{k+1 \pm \sqrt{(k+1)^2 - 4}}{2}$, $k = 1,2,..$, for which, vice versa, $\mathrm{Re}_{th} \to \infty$ in (14) and the HP flow linear instability can't be realized.

For p not equal to the noted above values $p_{\sqrt{k}}$, such an integral helicity can exponentially grow with time when realizing HP flow linear instability for $\mathrm{Re} > \mathrm{Re}_{th}$, where the threshold Reynolds number value Re_{th} is defined in (14) and (A.3).

The conclusions of the present work allow filling the well-known gap in the non-linear theory [7, 8], when instead of the linear exponential instability up to now it was necessary to consider the stage of the seed algebraic instability (where small initial disturbances can only locally in time grow tending to zero for $t \to \infty$).

Let us note also that it is reasonable to revise also the mentioned above problems on linear stability for the flat Couette flow and the flat Poisuille flow on the base of accounting of the obtained in the present work conclusion about possibility of the HP flow linear exponential instability due to the consideration of differing from the "normal" longitudinal quasi-periodic disturbances. Meanwhile, longitudinal disturbances quasi-periodicity is not by itself important but formation of it due to the longitudinal periods distinctions for different basic (in that case, radial) modes existing only for the non zero fluid viscosity.

Appendix

1. From (10) and (13) for $N = 2$ and *M=1* in (12) we get for $\lambda = \lambda_1 + i\lambda_2$:

$$\lambda_1 = -j_{1,1}^2 - 4\pi^2\beta^2 p^2 - \frac{1}{2}(a_1 \pm \frac{1}{\sqrt{2}} D_1^{1/2}) , \tag{A.1}$$

$$\lambda_2 = -2\pi\beta p P_{11}\,\mathrm{Re} - \frac{1}{2}(a_2 \pm \frac{1}{\sqrt{2}} D_2^{1/2}) , \tag{A.2}$$

where $D_1 = d_0^{1/2} + l$, $D_2 = d_0^{1/2} - l$, $l = a_1^2 - a_2^2 + 4c_1 \mathrm{Re}_1^2$

$a_1 = j_{1,2}^2 - j_{1,1}^2 + 4\pi^2\beta^2(1 - p^2)$, $a_2 = 2\pi\beta\,\mathrm{Re}(P_{22} - pP_{11})$,

$\mathrm{Re}_1^2 = \frac{\mathrm{Re}^2 P_{21}P_{12}p^2\beta^2}{(1-p)^2}$, $d_0 = l^2 + 4(a_1a_2 - 2\,\mathrm{Re}_1^2 d_1)^2$, $d_1 = -4S$,

$c_1 = -4Sctg\pi(p + \frac{1}{p})$, and S is defined in (14).

The condition $\lambda_1 > 0$ leads to the inequality

$$(a\,\mathrm{Re} + b)^2 > c + \frac{d}{\mathrm{Re}^2} \qquad \text{(A.3)}$$

where $a = \frac{4P_{21}P_{12}p^2\beta^2 S}{(1-p)^2}$, $b = \pi\beta(P_{22} - pP_{11})a_1$, $d = a_3^2(j_{1,1}^2 + 4\pi^2\beta^2 p^2)(j_{1,2}^2 + 4\pi^2\beta^2)$,

$a_3 = j_{1,2}^2 + j_{1,1}^2 + 4\pi^2\beta^2(1+p^2)$, $c = a_3^2\beta^2(\pi^2(P_{22} - pP_{11})^2 - 4\frac{P_{12}P_{21}p^2 S ctg\pi(p+\frac{1}{p})}{(1-p)^2})$.

2. In the limit $\mathrm{Re} >> 1$, inequality (A.3) with $c > 0$ is reduced to the inequality

$$\mathrm{Re} > \mathrm{Re}_{th}(\beta) = \frac{\sqrt{c} - b\cdot\frac{S}{|S|}}{a} \qquad \text{(A.4)}$$

In (A.4), the function $\mathrm{Re}_{th}(\beta)$ takes minimal value (given in (14)) for

$$\beta = \beta_0 = \frac{1}{2\pi}\left[\frac{A(j_{1,2}^2 + j_{1,1}^2) + B(j_{1,2}^2 - j_{1,1}^2)}{A(1+p^2) + B(1-p^2)}\right]^{1/2}, \qquad \text{(A.5)}$$

where A and B are defined in the main text (see (14)) for $A^2 > 0$.

3. On the neutral curve with $\lambda_1 = 0$ (i.e. when equality $\mathrm{Re} = \mathrm{Re}_{th}(\beta, p)$ in (A.4) holds), the phase velocity V_β / V_{cp} has the form

$$V_\beta / V_{cp} = pP_{11} + P_{22} \pm \left(\frac{D_2^{1/2}}{2\sqrt{2}\pi\beta\,\mathrm{Re}}\right)_{\beta=\beta_{1,2}} \qquad \text{(A.6)}$$

where $\beta = \beta_{1,2}(\mathrm{Re}, p)$ corresponds to replacing of inequality in (A.4) by equality. For such a replacement in (A.4), we get a quadratic equation with respect to β. Its solution is

$$\beta = \beta_{1,2} = \frac{\mathrm{Re} \pm \sqrt{\mathrm{Re}^2 - \mathrm{Re}_{th}^2}}{2\pi^2\delta_1} \qquad \text{(A.7)}$$

for $\mathrm{Re} \geq \mathrm{Re}_{th}$, where Re_{th} from (14), when $\beta_{1,2} = \beta_0$ for $\mathrm{Re} = \mathrm{Re}_{th}$, and $\delta_1 = \pi((p^2+1)\sqrt{1-4S\delta} - (1-p^2)\frac{Sb_1}{|Sb_1|})\frac{|b_1|(1-p)^2}{|S|\,p^2P_{12}P_{21}}$, $b_1 = P_{22} - pP_{11}$, $\delta = \frac{p^2P_{21}P_{12}}{(1-p)^2\pi^2 b_1^2} ctg\pi(p + \frac{1}{p})$.

4.

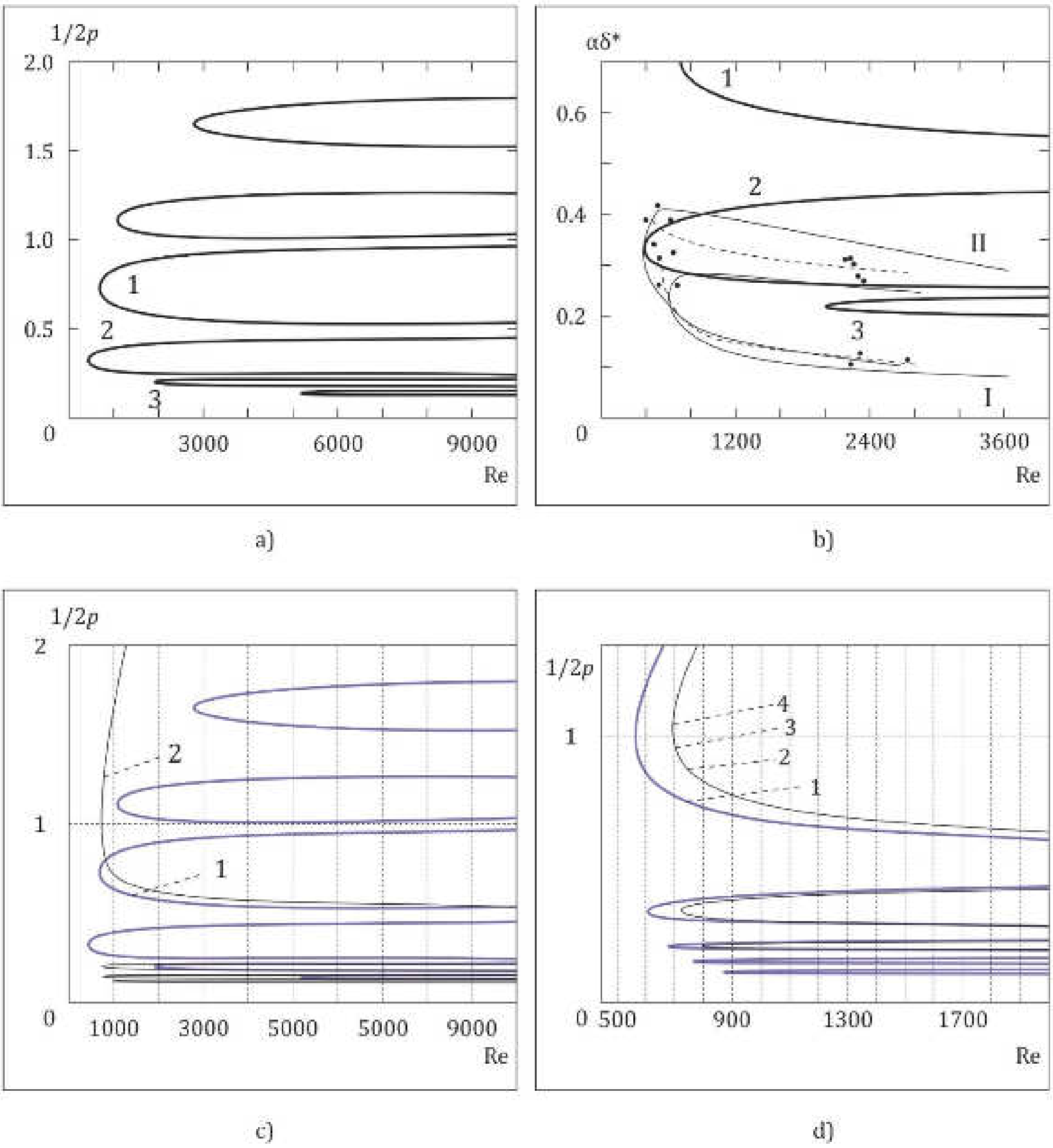

Figure 1. a) Family of the six curves of neutral stability (with $\lambda_1 = 0$), according to (14) and (A.5); the instability regions bounded by lines 2 and 3, respectively, correspond to $\beta_0 = 0.463$ (at *p=1.527*) and $\beta_0 = 1.099$ (at *p=2.239*). b) Curves 1-3 from panel a) in a magnified scale together with Fig.12 from [31] under the condition of the coincidence of the dimensionless parameters *1/2p=*$\alpha\delta^*$, where α is the wavenumber of the disturbance, and δ^* is the shift of the boundary layer in the flow around the thin plate [31]. The points and dashed curves correspond to the experiment reported in [31], and thin solid curves I and II are Shlihting and Lin theories, respectively. c) For *N=2*, Galerkin approximation (blue, labeled by 1) and enegy theory (black, labeled by 2), integer values of *p* are excluded from calculations; d) Energy theory for *N=2, 10, 100, 600* (numbered 1 to 4 respectively for one set of curves).

5. Table of values Re_{th} and β_0, obtained by (A.3) and (14) for p, corresponding to the local minima Re_{th} in the approximate formula (14)

p	β_0 (from (A.3))	Re_{th} (from (A.3))	β_0 (from (A.5))	Re_{th} (from (14))
1,527	0,471	448,455	0,463	442,278
0,674	1,124	680,307	1,101	678,482
0,447	1,368	1095,455	1,358	1093,824
2,239	1,100	1983,171	1,099	1981,838
2,791	0,220	13095,398	0,219	13095,285
0,359	1,114	23816,499	1,114	23816,488

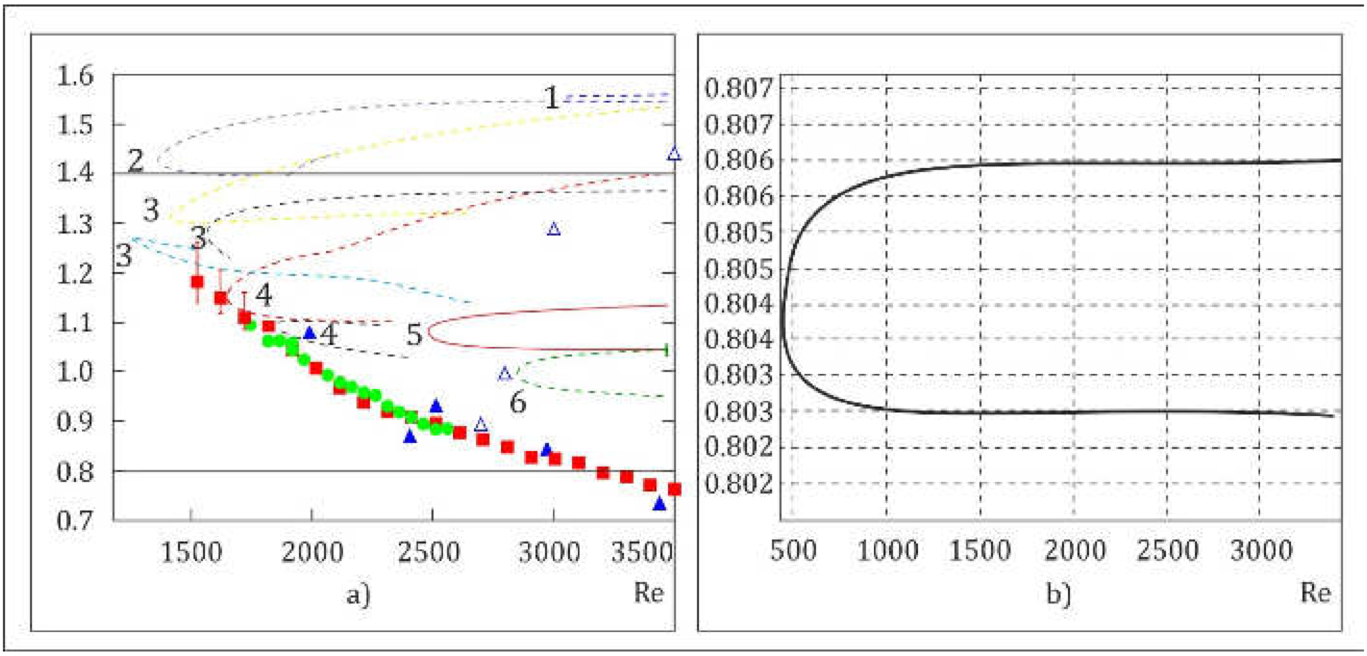

Figure 2. a) Results of the experiment [28] (for the velocity of the rear edge of a turbulent spot, blue rectangles, and for the velocity of the fore edge, light triangles) and [30] (rectangles and circles, for the velocity of the rear edge). The phase velocity of the wave solutions [8] (numbers 1-5 denote the level of azimuthal symmetry of the corresponding travelling wave). The velocities are normalized by the stream in the pipe volumetric/average velocity. Also the result of calculations of the phase velocity according to (A.6) is given. The top "straight" line corresponds to sign plus in (A.6), and the bottom one to sign minus for $p \approx 1.53$, β =0,471 (that corresponds to the absolute minimum Re_{th} =448.5 according to (A.3). b) Zoomed representation of the bottom "straight" line from Fig.2,a according to (A.6) (the upper brunch on Fig.2,b corresponds to sign plus in (A.7), and the lower to sign minus).

Statement on ethical issues

Research involving people and/or animals is in full compliance with current national and international ethical standards.

Acknowledgements

We are grateful to S.I. Anisimov, G.S. Golitsin, V. P. Goncharov, E.A. Novikov, and N.A. Inogamov for useful comments and interest to the work.

Conflict of interest

None declared.

Author contributions

All authors prepared the manuscript and analyzed the data, S.G.C. drafted the manuscript. All authors read the ICMJE criteria for authorship and approved the final manuscript.

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