REPORT

Submitted: 29.12.2020; Accepted: 23.01.2020; Published online: 21.05.2021

Development of the single ventricle heart mathematical model based on the equation of forced oscillations

Aleksandr P. Chernyaev[1]*, Elena R. Pavlyukova[2], Alla Yu. Meyerson[3]

[1] Moscow Institute of Physics and Technology
Russia, 141701, Moscow region, Dolgoprudny, Institutsky per., 9
[2] Kotel'nikov Institute of Radio Engineering and Electronics of RAS
Russia, 125009, Moscow, Mokhovaya str. 11, build. 7
[3] Plekhanov Russian University of Economics
Russia, 117997, Moscow, Stremyaniy per, 36
* Corresponding author:
phone: +7 (903) 558-85-69
email: chernyaev40@yandex.ru

Abstract

Not only protozoa being the simplest organisms have a single ventricle heart. Artificial heart systems used in cardiac surgery can be also of the single-ventricle design. Besides, there is a "single-ventricle heart condition", i.e. a complex of defects in the structure of the heart that may occur in some human individuals and that may be found with different occurrence rate and severity. This defect should be attributed to an uneven development and malformation of some individual heart segments in the prenatal period. Single Ventricle belongs to the category of congenital heart defects. The latter greatly increases the scientific interest in studying the features of mathematical models of the single-ventricle heart condition. Herein, we offer our study on some issues of the smoothness of sewing of the exact solutions of a mathematical model of a single-ventricle heart based on the equation of forced oscillations.

Keywords

Active heart ventricle, Passive heart ventricle, Equations for free and forced oscillations, Artificial single-ventricle heart, Congenital defect

Imprint

Aleksandr P. Chernyaev, Elena R. Pavlyukova, Alla Yu. Meyerson. Development of the single ventricle heart mathematical model based on the equation of forced oscillations. Cardiometry; Issue 18; May 2021; p.56-59; DOI: 10.18137/cardiometry.2021.18.5659; Available from: http://www.cardiometry.net/issues/no18-may-2021/heart-mathematical-model

Introduction

The topicality of research on exact solutions of the single-ventricle heart mathematical models is supported by some considerations as stated below herein. Protozoa as the simplest organisms have a single-ventricle heart anatomy. Artificial single-ventricle heart systems are known to be used in cardiac surgery. Sometimes it occurs that due to an uneven development in the prenatal period, one of the ventricles of the heart is formed in the proper manner, to the degree of its full functionality, while the other either is fully absent, or is malformed so that it is not capable to normally maintain the proper blood circulation. Considering these possible variants, we can generally define the actual heart condition either as the heart with a single ventricle, or as the heart with the main, primary, ventricle receiving blood from both atria, and "the auxiliary, secondary," ventricle receiving blood from the main one. Different options of the development–underdevelopment of the ventricles have their different effects on the clinical sign picture, result in more or less acute problems with blood circulation, pathological consequences of the heart failure and the patient's condition. They also determine various methods of treatment that are intensively practicized in Germany (Single Ventricle).

Based on the studies on the four-ventricle heart [1-6], the formulation of the problem of the single-ventricle heart performance has been offered [7]. In [8], in addition to the relevant reference sources treating various mathematical models of the heart, it has been proven that the solution of the problem of the single-ventricle heart performance is reduced to finding solutions to the equation of forced oscillations [9-14]. A set of solutions to the equation of forced oscillations is divided into classes depending on the roots of the characteristic equation. In these different modes, we study the smoothness of sewing of the exact solutions if amplitude and a frequency of the external pressure change at some point of time.

1. Simplified mathematical model of a single-ventricle heart

In [7], in addition to the problem of the single-ventricle heart performance, an equivalent electrical diagram of the circulatory system with one active ventricle is given. The hemodynamic model in [7] consists of one active and one passive chamber, respectively. This model describes the hemodynamic system found in protozoa. Artificial hearts used in cardiac surgery can also be of the single-ventricle design type. Congenital heart defects (Single Ventricle) as a first approximation can also be described in a similar way. The hemodynamic model in [7] includes the following four equations:

$$I_1 V_1''(t) + R_1 V_1'(t) + C_1^{-1} = P_1(t) - F(t) \quad (1.1)$$

$$I_2 V_2''(t) + R_2 V_2'(t) + C_2^{-1} V_2(t) = P_2(t) \quad (1.2)$$

$$R_{1,2} V_1'(t) = P_2(t) - P_1(t) \quad (1.3)$$

$$R_{2,1} V_2'(t) = P_1(t) - P_2(t) \quad (1.4)$$

In (1.1) – (1.4) $V_1(t)$, $V_2(t)$ are the volumes of the first active and the second passive chambers to be determined; $P_1(t)$, $P_2(t)$ are the pressures in these chambers; F(t) is an additional external pressure produced by the active wall of the first chamber; I_1, I_2 are coefficients of inertia in the chambers; C_1, C_2 are expansion coefficients; R_1, R_2 are chamber resistance coefficients, and $R_{1,2}$ is the inter-chamber flow resistance coefficient. From (1.3) and (1.4) immediately follows

$$V_1'(t) = -V_2'(t) \quad \Rightarrow V_1(t) + V_2(t) = V_0 = const.$$

2. Reduction of the single-ventricle heart model to a single equation of forced oscillations

It is shown in [8 – 10] that the problem (1.1) – (1.4) is reduced to finding a solution to a single equation of forced oscillations

$$\begin{aligned}&(I_1 + I_2) V_1''(t) + \\ &+ (R_1 + R_2 + R_{1,2}) V_1'(t) + \\ &+ (C_1^{-1} + C_2^{-1}) V_1(t) = \\ &= C_2^{-1} V_0 - F(t)\end{aligned} \quad (2.1)$$

Equation (2.1) has three different solution modes depending on whether the roots of the characteristic equation

$$\begin{aligned}&(I_1 + I_2)\lambda^2 + \\ &+ (R_1 + R_2 + R_{1,2})\lambda + \\ &+ (C_1^{-1} + C_2^{-1}) = 0\end{aligned} \quad (2.2)$$

are different real numbers, coinciding the real ones, or complex conjugates.

We describe all solutions of equation (2.1). Let us introduce designations as follows:

$$I = I_1 + I_2,\ R = R_1 + R_2 + R_{1,2},\ 1/C = 1/C_1 + 1/C_2,$$

$$If(t) = V_0/C_2 - F(t),\ V(t) = V_1(t) \quad (2.3)$$

Equations (2.1) and (2.2), taking into account (2.3), will have the following form:

$$IV''(t) + RV'(t) + C^{-1}V(t) = If(t), \quad (2.4)$$

$$I\lambda^2 + R\lambda + C^{-1} = 0. \quad (2.5)$$

3. Three types of exact solutions to the forced oscillation equation with integral representation of partial solutions

We describe three types of exact solutions to equation (2.4) in other form than it is given in [11]. Let us introduce designations as follows:

$$a = \frac{R}{I}, b = \frac{1}{CI}, y = V \quad (3.1)$$

Due to (3.1), equation (2.4) is transformed to the following form:

$$y''(t) + ay'(t) + by(t) = f(t), \quad (3.2)$$

Referring to 2.36 (a) [11, p. 376] in case of $\lambda_2 = a_2 - 4b > 0$, $\lambda > 0$, the general solution to equation (3.2) has the following form:

$$y(t) = \frac{2}{\lambda}\int_{t_0}^{t} f(s) e^{\frac{a}{2}(s-t)} sh\frac{\lambda}{2}(t-s)ds + Ae^{\left(-\frac{a}{2}-\frac{\lambda}{2}\right)t} + Be^{\left(-\frac{a}{2}+\frac{\lambda}{2}\right)t}, \quad (3.3)$$

A = const, B = const

Since, for the roots of characteristic equation (2.5), the equalities as given below hold true:

$$\lambda_1 = -\frac{a}{2} - \frac{\lambda}{2}, \lambda_2 = -\frac{a}{2} + \frac{\lambda}{2}, \lambda = \lambda_2 - \lambda_1,$$

and formula (3.3) is transformed to the following form:

$$V(t) = \frac{1}{\lambda_2 - \lambda_1}\int_{t_0}^{t} f(s)\left[e^{\lambda_2(t-s)} - e^{\lambda_1(t-s)}\right]ds + Ae^{\lambda_1 t} + Be^{\lambda_2 t}, \quad (3.4)$$

A = const, B = const

In case of $\mu^2 = 4b - a^2 > 0$, $\mu > 0$, the general solution to equation (3.2) according to 2.36 (b) [11, p. 376] will have the form:

$$y(t) = \frac{2}{\mu}\int_{t_0}^{t} f(s) e^{\frac{a}{2}(s-t)} \sin\frac{\mu}{2}(t-s)ds + Ae^{-\frac{a}{2}t}\cos\frac{\mu}{2}t + Be^{-\frac{a}{2}t}\sin\frac{\mu}{2}t. \quad (3.5)$$

A = const, B = const

In this case, for the roots of characteristic equation (2.5), the equalities hold true as given below:

$$\lambda_1 = -\frac{a}{2} - \frac{i\mu}{2}, \lambda_2 = -\frac{a}{2} + \frac{i\mu}{2}, i\mu = \lambda_2 - \lambda_1$$

and formula (3.5) is transformed to form (3.4).

In case of $4b = a^2$, according to 2.36 (b) [11, p. 376] the general solution to equation (3.2) will be the following:

$$y(t) = \int_{t_0}^{t} f(s)(t-s)e^{\frac{a}{2}(s-t)}ds + Ae^{-\frac{a}{2}t}t + Be^{-\frac{a}{2}t} \quad (3.6)$$

A = const, B = const

Since the formula $\lambda_0 = \lambda_1 = \lambda_2 = -a/2$ is valid for the two-fold root of the characteristic equation, formula (3.6) can be written as

$$V(t) = \int_{t_0}^{t} f(s)(t-s)e^{\lambda_0(t-s)}ds + Ae^{\lambda_0 t}t + Be^{\lambda_0 t} \quad (3.7)$$

A = const, B = const

Note that in formulas (3.3), (3.4), (3.5), (3.6) and (3.7) the first summand is a particular solution to equation (2.1), and it is represented by an integral, which depends on parameter t . The second and third summands represent the general solution to the equation of free oscillations [11, p. 375]

$$IV''(t)+RV'(t)+C^{-1}V(t)=0\,, \qquad (3.8)$$

which differs from the equation of forced oscillations (2.4) in the zero right part only.

Periodic solutions to case $\mu^2 = 4b-a^2>0$, $\mu>0$ are well studied [12 -14]. In the examples given in [8-10], the particular solutions of equation (2.4) are expressed by trigonometric functions, since the integrals in the integral representations of the particular solutions are taken explicitly.

4. Method for obtaining estimates of the order of smoothness at the point of change of amplitudes and frequencies of additional external pressure produced by the wall of the first chamber

Let us assume that an additional external pressure produced by the active wall of the first chamber at some point in time $t_1 > t_0$ change the amplitude and frequency, that is,

$$F(t)=\begin{cases} a_0\sin(\omega_0 t+\varphi_0),\ t_0\le t\le t_1; \\ a_1\sin(\omega_1 t+\varphi_1),\ t_1<t<+\infty. \end{cases} \qquad (4.1)$$

We impose a natural condition for the continuity of function (4.1) at point t_1:

$$a_0\sin(\omega_0 t_1+\varphi_0)=a_1\sin(\omega_1 t_1+\varphi_1) \qquad (4.2)$$

For the time derivative of function (4.1), we can have

$$\dot F(t)=\begin{cases} a_0\omega_0\cos(\omega_0 t+\varphi_0),\ t_0<t<t_1; \\ a_1\omega_1\cos(\omega_1 t+\varphi_1),\ t_1<t<+\infty. \end{cases} \qquad (4.3)$$

We assume that function (4.3) at point t_1 is not smooth, and that means that the following inequality is valid:

$$a_0\omega_0\cos(\omega_0 t_1+\varphi_0)\ne a_1\omega_1\cos(\omega_1 t+\varphi_1) \qquad (4.4)$$

We now show the smoothness of the equation (2.4) solution at point t_1, and also estimate the degree of this smoothness. Let us consider formulas (3.4) and (3.7). Let us compare the solutions according to formulas (3.4) and (3.7) with the initial disturbance specified by formulas (4.1) and

$$\hat F(t)=a_0\sin(\omega_0 t+\varphi_0),\ t_0<t<+\infty \qquad (4.5)$$

Thus, for smoothness at point t_1, we need to consider the difference

$$V(t)-\hat V(t) \qquad (4.6)$$

From the fourth formula (2.3) it follows that

$$f(t)-\hat f(t)=I^{-1}[\hat F(t)-F(t)] \qquad (4.7)$$

It follows from (3.4) and (3.7) that only the particular solutions to the equation of forced oscillations need to be compared, because the second and third summands are the same and therefore are reduced. Further, from (4.7), (3.4) and (3.7), since the right part of (4.7) differs from zero only by (t_1, t) , then the smoothness should be considered as given below:

$$J_1(t)=\int_{t_1}^{t} f(s)\left[a_0\sin(\omega_0 s+\varphi_0)-a_1\sin(\omega_1 s+\varphi_1)\right]\left[e^{\lambda_2(t-s)}-e^{\lambda_1(t-s)}\right]ds\,, \qquad (4.8)$$

$$J_2(t)=\int_{t_1}^{t} f(s)\left[a_0\sin(\omega_0 s+\varphi_0)-a_1\sin(\omega_1 s+\varphi_1)\right](t-s)e^{\lambda_0(t-s)}ds\,. \qquad (4.9)$$

We omitted the non-zero constant multipliers before the integrals. To differentiate these integrals, we use the Leibniz formula [15]:

$$J_1'(t)=\int_{t_1}^{t}\left[a_0\sin(\omega_0 s+\varphi_0)-a_1\sin(\omega_1 s+\varphi_1)\right]\left[\lambda_2 e^{\lambda_2(t-s)}-\lambda_1 e^{\lambda_1(t-s)}\right]ds\,, \qquad (4.10)$$

$$J_2'(t)=\int_{t_1}^{t}\left[a_0\sin(\omega_0 s+\varphi_0)-a_1\sin(\omega_1 s+\varphi_1)\right]\left[e^{\lambda_0(t-s)}+\lambda_0(t-s)e^{\lambda_0(t-s)}\right]ds\,. \qquad (4.11)$$

The Leibniz formula consists of three summands [15], among which only one remains in (4.10) and (4.11). This is because the lower limit in (4.8) and (4.9) is constant, and substituting the upper limit in the integrand for the integration variable results in zero. It is quite clear that in case of $t \to t_1$ both integrals (4.10) and (4.11) tend to zero that already indicates the desired smoothness. However, in addition, the integrands in (4.10) and (4.11) tend to zero at $t \to t_1$ that follows from (4.2). And this indicates increased smoothness. To obtain an estimate of the tendency to zero in (4.10) and (4.11), we apply the mean value theorem to the integrals in the right parts of (4.10) and (4.11):

$$J_1'(t)=[a_0\sin(\omega_0\tau+\varphi_0)-a_1\sin(\omega_1\tau+\varphi_1)]\int_{t_1}^{t}\left[\lambda_2 e^{\lambda_2(t-s)}-\lambda_1 e^{\lambda_1(t-s)}\right]ds=$$
$$=[a_0\sin(\omega_0\tau+\varphi_0)-a_1\sin(\omega_1\tau+\varphi_1)]\left[e^{\lambda_1(t-s)}-e^{\lambda_2(t-s)}\right]\Big|_{t_1}^{t}=$$
$$=[a_0\sin(\omega_0\tau+\varphi_0)-a_1\sin(\omega_1\tau+\varphi_1)]\left[e^{\lambda_2(t-t_1)}-e^{\lambda_1(t-t_1)}\right], \qquad (4.12)$$

$$J_2'(t)=\left[a_0\sin(\omega_0\tau+\varphi_0)-a_1\sin(\omega_1\tau+\varphi_1)\right]\int_{t_1}^{t}\left[e^{\lambda_0(t-s)}+\lambda_0(t-s)e^{\lambda_0(t-s)}\right]ds=$$
$$=\left[a_0\sin(\omega_0\tau+\varphi_0)-a_1\sin(\omega_1\tau+\varphi_1)\right]\int_{t_1}^{t}\left[e^{\lambda_0(t-s)}+\lambda_0(t-s)e^{\lambda_0(t-s)}\right]ds=$$
$$=\left[a_0\sin(\omega_0\tau+\varphi_0)-a_1\sin(\omega_1\tau+\varphi_1)\right](t-t_1)e^{\lambda_0(t-t_1)}. \qquad (4.13)$$

In (4.12) and (4.13) we have $\tau\in(t_1,t)$. Estimates (4.12) and (4.13) show the degree of smoothness in the solution to the equation of forced oscillations at the point of change in the amplitude and frequency of the additional external pressure produced by the active wall of the first chamber.

Conclusions

It should be mentioned that the single-ventricle hearts are found not only in protozoa as the simplest living organisms, but also as the design version of artificial single-ventricle hearts employed in cardiac surgery. Known are some congenital heart defects, which are referred to as the “single-ventricle heart condition" and which are the result from an uneven development or malformation of some individual heart segments in the prenatal period (a Single Ventricle Condition). A single-ventricle heart is easier to control than a multi-ventricle one. It gains even more attraction due to the exact solutions presented herein. Examples in [8 -10] describe the modes of the single-ventricle heart performance, when the amplitude and frequency demonstrate no changes with time. The processes of the single-ventricle heart performance, when the amplitude and frequency change at some point in time, are described by changing the amplitude and frequency of the additional external pressure produced by the active wall of the chamber. It has been shown that a non-smooth change in the external pressure leads to a sufficiently smooth change in the solution to equation (2.4). The obtained estimates of the degree of smoothness at the point of inflection of the external pressure show the absence of rhythm interruptions. The success of this study is based on the obtained herein solutions to equation (2.4) with an integral representation of the particular solution to the equation.

Statement on ethical issues

Research involving people and/or animals is in full compliance with current national and international ethical standards.

Conflict of interest

None declared.

Author contributions

The authors read the ICMJE criteria for authorship and approved the final manuscript.

References

1. Kholodov AS. Some dynamic models of external respiration and blood circulation, taking into account their connectivity and substances transfer. Computer models and medicine progress. Moscow: Science, 2001, 392 p. P. 127-163 [in Russian]

2. Simakov SS. Numerical study of the dynamics of systemic blood flow during blood loss. Information technology modeling and management. 2006;8(33):931-8. [in Russian]

3. Simakov SS, Kholodov AS. Numerical study of oxygen content in human blood at low-frequency effects. Math modeling. 2008;20(4):87-102. [in Russian]

4. Vasilevsky YuV, Salamatova VYu, Simakov SS. On the elasticity of vessels in one-dimensional hemodynamic models. Magazine computing mathematics and mathematical physics. 2015;55(9):1599-610.

5. Bessonov N, et al. Methods of Blood Flow Modelling. Math. Model. Nat. Phenom. 2016;11(1):1–25.

6. Simakov SS. Modern methods of mathematical modeling of blood flow using the averaged models. Computer research and modeling. 2018;10(5):581-604.

7. Kalyabin GA. Application of mathematical analysis to the description of the processes of reparation of the myocardial infarction and forecasting of cardiological diseases: studies. Benefit. - Moscow: Rudn. 2008. - 144 p.

8. Chernyaev AP. Accurate solutions to the mathematical model of the single-chamber heart and the conditions of the rhythmic work of the cardiac chamber. IT - Technologies: Development and Applications. Collection of reports of the International Scientific and Technical Conference. Vladikavkaz, December 2019, pp. 163-171.

9. Chernyaev AP. Exact solutions of the mathematical model of a single–chamber heart based on the equations of a forsed oscillations. Analytical and numerical methods for solving of hydrodynamics, mathematical physics and biology problems. International conference dedicated to the 100 –th anniversary of K.I.Babenko. 26–29 August 2019. Pushino, Moscow Region. Proceedings. Moscow – 2019. P. 52–53.

10. Chernyaev AP. Basics of the hypothesis of the rhythmic work conditions of the cardiac chamber in some mathematical models. Radio electronics magazine. 2019. № 6. Available online: http://jre.cplire.ru/jre/jun19/11/text.pdf DOI 10.30898/1684-1719.2019.6.11

11. Kamke E. Handbook on ordinary differential equations. Moscow: Nauka. 1976. 576 p.

12. Pontryagin LS. Ordinary Differential Equations. Moscow: Nauka. 1974. 331 p.

13. Stepanov VV. Course of differential equations: Textbook for universities. Moscow: URSS Editorial, 2004. - 472 p.

14. Stoker J. Nonlinear fluctuations in mechanical and electrical systems. M.: Il. 1952. 265 p.

15. Chernyaev AP. Fourier rows. Integrals depending on the parameter and generalized functions. Lecture course. Moscow: MH-Press, 2004. - 149 p.